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Statistical Properties of Aggregation with Injection

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We study a generalized aggregation process in which charged particles diffuse and coalesce randomly on a lattice. For one-dimensional and mean-field models, we show that there exists a statistically-invariant steady state when randomly charged particles are continuously injected. The steady-state charge distribution obeys a power law with the exponent depending both on the type of the injection and on the spatial dimension. The response of the system to a perturbation (i.e., relaxation) is characterized by either a power law decay ($t^{-\beta}$, $\beta \leqslant 1$) or a compressed exponential decay [$\exp(-t^{\alpha})$, $\alpha > 1$].

KEY WORDS: Aggregation; injection; power law distribution; stability; relaxation; stable distribution.

1. INTRODUCTION

Aggregation of diffusive particles shows peculiar statistical behavior due to its intrinsic irreversible nature. Recently much work on aggregation has involved the notion of fractals.(1,2) It is now clear that complicated geometrical objects, characterized by fractional power laws, can be produced by simple aggregation and diffusion rules.(3)

In aggregation systems, fractional power laws can be found in the spatial as well as the temporal behavior of the mass-distribution function. In a closed system, the total number of particles decays as a power law with respect to time.(3–5) A statistically-invariant steady state can be realized by continuous injetion of small particles. In this case the number of particles converges to a value determined by the balance of aggregation

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and injection rates.[3,6] The mass distribution which characterizes this steady state is also known to follow a power law.[7,8]

In general, power laws appear in the behavior of systems at thermal equilibrium near the critical point. However, in aggregation systems criticality often arises automatically. There are many phenomena in nature which show power law behavior in the absence of a control parameter;[2,9] remarkably, they all occur in irreversible dissipative systems. Since aggregation is a typical dissipative phenomenon, we expect that the study of the statistical properties of aggregation will contribute to the construction of a statistical physics of irreversible systems.

In this paper, we investigate the statistical properties of aggregation with injection. Some power laws are found for one-dimensional and mean-field cases, together with results from numerical simulations. In the following section the model is described and the basic equations are derived. The solution for the case of no injection is given in the third section. In the fourth section we study the statistical properties of the steady state realized by continuous injection. We show that there exists a stable steady state which obeys a robust power law distribution. Finally, in the fifth section we examine the response of the steady-state power law to an initial perturbation. We show that the power law is stable and that any perturbation quickly relaxes to the steady state, following either a power law decay or a compressed exponential decay, depending on the type of injection and the dimensionality of the space.

2. THE MODEL AND THE BASIC EQUATIONS

We consider the aggregation process in discretized space and time (one dimension and mean field). On every site there is at most one particle. If more than two particles happen to hop onto one site, they immediately coalesce into a single particle with the charge of the aggregate equal to the sum of the charges of the two incident particles. Let $m(j, t)$ be the charge of the particle on site j at the tth time step. The aggregation process can be represented by the following stochastic equation for $m(j, t)$:

$$m(j, t+1) = \sum_k W_{jk}(t)\, m(k, t) + I(j, t) \tag{1}$$

where $I(j, t)$ denotes the charge injected at the jth site at time t, and $W_{jk}(t)$ is a random variable which is equal to 1 when the particle on the kth site jumps on the jth site and is equal to zero otherwise. Since one particle cannot go to two different sites in a single time step, $W_{jk}(t)$ must be normalized: $\sum_j W_{jk}(t) = 1$. In the following analysis we shall consider two simple cases:

(A) $W_{jj}(t) = 1$ with probability 1/2, $W_{jj-1}(t) = 1$ with probability 1/2, and $W_{jk}(t) = 0$ for $k \neq j$, $j-1$.

(B) $W_{jk}(t) = 1$ with probability $1/N$, where N is the total number of sites, which will tend to infinity in our analysis.

Case (A) with periodic boundary conditions corresponds to the situation of aggregating Brownian particles in one dimension, if we observe the system from a coordinate which moves with a constant mean velocity of 1/2. Case (B) corresponds to the mean-field limit. For injection, we consider two cases: (I) independent random injection, and (II) pair-creation injection. In the random-injection case, we inject random charges independently at every site at every time step. In the pair-creation case, we inject pairs of positive and negative charges simultaneously, but randomly, on pairs of adjacent sites [For example, a pair injection onto the jth and $(j+1)$th sites is given by $I(j, t) = -I(j+1, t)$.] This special case is particularly interesting because, as we shall explain in Section 6, it sheds light on the problem of the critical dimension of our model. The injection for each time step is completed when all sites have, on average, one injected particle.

In the one-dimensional case (A), the space-time trajectories of particles plotted in oblique coordinates give the riverlike pattern shown in Fig. 1. Actually this pattern is identical to Scheidegger's river model on a slope.[10] The charge of a particle at (j, t) is equal to the sum of the charges injected

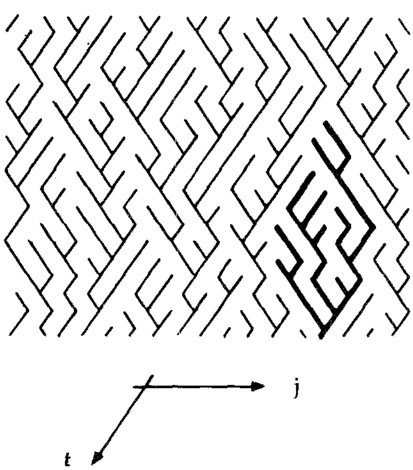

Fig. 1. An example of a space-time configuration of sticky random walkers in one-dimension with injection.

over the branches of the river corresponding to the drainage basin of the point (j, t).

The time evolution of the charge distribution can be analyzed by introducing the r-body characteristic function,

$$Z_r(\rho, t) = \left\langle \exp\left[i\rho \sum_{j=1}^{r} m(j, t) \right] \right\rangle \tag{2}$$

where $\langle \cdots \rangle$ denotes the average over all realizations of $\{W_{jk}(t)\}$ and $\{I(j, t)\}$. In case (A), with independent random injection, we have the following evolution equation,[8] for $Z_r(\rho, t)$ from Eq. (1),

$$\begin{aligned} Z_r(\rho, t+1) &= \left\langle \exp\left\{ i\rho \sum_{j=1}^{r} \sum_{k} W_{jk} m(k, n) + i\rho \sum_{j=1}^{r} I(j, t) \right\} \right\rangle \\ &= \frac{\langle \exp[i\rho \sum_{j=1}^{r} I(j, t)] \rangle}{4} \left\{ \left\langle \exp\left[i\rho \sum_{j=0}^{r} m(j, t) \right] \right\rangle \right. \\ &\quad + \left\langle \exp\left[i\rho \sum_{j=0}^{r-1} m(j, t) \right] \right\rangle + \left\langle \exp\left[i\rho \sum_{j=1}^{r} m(j, t) \right] \right\rangle \\ &\quad \left. + \left\langle \exp\left[i\rho \sum_{j=1}^{r-1} m(j, t) \right] \right\rangle \right\} \end{aligned} \tag{3}$$

Using translational invariance, we have

$$Z_r(\rho, t+1) = \tfrac{1}{4}\Phi(\rho)^r \left[Z_{r+1}(\rho, t) + 2Z_r(\rho, t) + Z_{r-1}(\rho, t) \right] \tag{4}$$

where $\Phi(\rho) \equiv \langle \exp[i\rho I(j, t)] \rangle$ is the characteristic function for the injection process, which can be expanded as

$$\Phi(\rho) = 1 + i\langle I \rangle \rho - \langle I^2 \rangle \rho^2/2 + \cdots \tag{5}$$

Equation (4) gives a set of linear equations for $r = 1, 2, \ldots, N-1$. By definition,

$$Z_0(\rho, t) = 1 \tag{6}$$

and for $r = N$, $Z_N(\rho, t)$ is the characteristic function for the total charge of the system, i.e.,

$$Z_N(\rho, t) = \Phi(\rho)^{tN} \cdot Z_N(\rho, 0) \tag{7}$$

In the case of pair-creation injection, positive and negative injections cancel for $j = 2$ to $r-1$ in Eq. (3). We have the following set of evolution equations

$$Z_1(\rho, t+1) = \tfrac{1}{4}\Phi(\rho)\{ Z_2(\rho, t) + 2Z_1(\rho, t) + 1 \} \tag{8a}$$

and for $r = 2, 3, ..., N-1$

$$Z_r(\rho, t+1) = \tfrac{1}{4}\Phi(\rho)^2 \{Z_{r+1}(\rho, t) + 2Z_r(\rho, t) + Z_{r-1}(\rho, t)\} \tag{8b}$$

The evolution equation for the mean-field case (B) can be written as

$$Z_1(\rho, t+1) = \Phi(\rho) \sum_{r=0}^{N} a_r Z_1(\rho, t)^r \tag{9}$$

where

$$a_r \equiv \binom{N}{r}\left(\frac{1}{N}\right)^r \left(1 - \frac{1}{N}\right)^{N-r}$$

This equation is also valid for pair-creation injection because all sites can be regarded as nearest neighbors in this case. In the limit $N \to \infty$ the right-hand side of Eq. (9) averages to an exponential function, namely

$$Z_1(\rho, t+1) = \Phi(\rho) \cdot \exp[Z_1(\rho, t) - 1] \tag{10}$$

This is the basic equation for the mean-field case (B).

3. THE SOLUTION FOR NO INJECTION

In order to see the effect of random aggregation, we first consider the classical case without injection. First, we solve the one-dimensional case. The evolution equation for $Z_r(\rho, t)$ is given by Eq. (4) with $\Phi(\rho) = \langle \exp[i\rho 0] \rangle = 1$,

$$Z_r(\rho, t+1) = \tfrac{1}{4}\{Z_{r+1}(\rho, t) + 2Z_r(\rho, t) + Z_{r-1}(\rho, t)\} \tag{11}$$

This equation can be considered a diffusion equation in (r, t) space. Due to the linearity of Eq. (11), the solution is given as

$$Z_r(\rho, t) = \sum_{r'=-t+r}^{t+r} G_{r,r'}(t) \cdot Z_{r'}(\rho, 0) \tag{12}$$

where $G_{r,r'}(t)$ is the Green function for Eq. (11),

$$G_{r,r'}(t) = \frac{1}{4^t}\binom{2t}{t-r-r'} \tag{13}$$

and $Z_r(\rho, t)$ for negative r is defined as

$$Z_r(\rho, t) = 2 - Z_{-r}(\rho, t) \tag{14}$$

which ensures the boundary condition (6). $G_{r,r'}(t)$ gives the probability that a random walker starting at $(r', 0)$ reaches (r, t). For large t, Eq. (13) can be approximated by a Gaussian function:

$$G_{r,r'}(t) \cong \frac{1}{\sqrt{\pi t}} e^{-(r-r')^2/t}, \qquad t \to \infty \tag{15}$$

where the symbol $\cong$ means equivalence up to first order. Substituting Eq. (15) into Eq. (12), and taking the continuum limit for r, we get

$$\begin{aligned} Z_r(\rho, t) &\cong \int_0^\infty dr' \frac{1}{\sqrt{\pi t}} e^{-(r-r')^2/t} Z_{r'}(\rho, 0) \\ &\quad + \int_0^\infty dr' \frac{1}{\sqrt{\pi t}} e^{-(r+r')^2/t} [2 - Z_{r'}(\rho, 0)] \\ &= 1 - r \frac{2}{\sqrt{\pi t}} \int_0^\infty dr' \frac{2r'}{t} e^{-(r')^2/t} [1 - Z_{r'}(\rho, 0)] \end{aligned} \tag{16}$$

where we have expanded the exponential function to first order in $r/\sqrt{t}$.

Let us solve for the decay of particle number with the initial condition that every site is occupied by a particle having unit charge. The initial condition is given by $Z_r(\rho, 0) = e^{i\rho r}$. The evolution of the charge distribution is obtained by Fourier inversion of Eq. (16),

$$\begin{aligned} p_r(m, t) &\cong \frac{1}{2\pi} \int d\rho\, e^{-i\rho m} Z_r(\rho, t) \\ &= \left(1 - \frac{2r}{\sqrt{\pi t}}\right) \delta(m) + \frac{2r}{\sqrt{\pi t}} \frac{2m}{t} e^{-m^2/t} \end{aligned} \tag{17}$$

Here $p_r(m, t)$ denotes the probability that the sum of the charges at r successive sites is m, at time step t. From Eq. (17) we know that the charge distribution for large m decays following a Gaussian function and the particle number decreases for large t as

$$N \cdot p_1(m \neq 0, t) \sim \frac{2N}{\sqrt{\pi t}}, \qquad t \to \infty \tag{18}$$

where the symbol $\sim$ means proportional up to first order. This result is consistent with those of previous authors[5] who treated the same problem by different methods.

Next, we consider the mean-field case (B). For large t we can assume that $Z_1(\rho, t) = 1$ because the existence probability for particles should vanish as $t \to \infty$. From Eq. (10), with $\Phi(\rho) = 1$ we have the following evolution equation:

$$Z_1(\rho, t+1) - Z_1(\rho, t) = \tfrac{1}{2}[Z_1(\rho, t) - 1]^2 + \cdots \tag{19}$$

By taking the continuum limit with respect to t, we obtain $Z_1(\rho, t)$,

$$Z_1(\rho, t) \cong 1 - \frac{2[1 - Z_1(\rho, 0)]}{t[1 - Z_1(\rho, 0)] + 2} \tag{20}$$

The decay in particle number can be calculated in the same way and we get

$$p_1(m \neq 0, t) \sim \frac{2}{t}, \qquad t \to \infty \tag{21}$$

for the same initial conditions as in the one-dimensional case. Notice that in the one-dimensional case, Eq. (18), the decay is slower than in the mean-field case, Eq. (21), due to the effect of spatial restriction.

4. THE STEADY STATE IN THE CASE OF INJECTION

In this section we study the steady state realized by continuous and statistically homogeneous injection. We divide this section into three subsections. We first obtain the steady-state charge distribution and then show its uniqueness and stability. Finally, the spatial correlation is analyzed for the one-dimensional case. The system size N is assumed to be *infinite* throughout.

4.1. THE CHARGE DISTRIBUTION

The charge distribution in the one-dimensional case with random independent injection can be obtained from the steady-state solution of Eq. (4). We have the following set of linear equations for $Z_r(\rho)$, $r = 1, 2, 3,\ldots,$ with the boundary condition (6):

$$Z_{r+1}(\rho) + [2 - 4\Phi(\rho)^{-r}]\, Z_r(\rho) + Z_{r-1}(\rho) = 0 \tag{22}$$

Dividing both sides of Eq. (22) by $Z_r(\rho)$, we get a recurrence relation for $Z_r(\rho)/Z_{r-1}(\rho)$,

$$\frac{Z_r(\rho)}{Z_{r-1}(\rho)} = \frac{1}{4\Phi(\rho)^{-r} - 2 - Z_{r+1}(\rho)/Z_r(\rho)} \tag{22'}$$

Therefore, $Z_1(\rho)$ is given by the following continued fraction:

$$Z_1(\rho)=\cfrac{1}{4\Phi(\rho)^{-1}-2-\cfrac{1}{4\Phi(\rho)^{-2}-2-\cfrac{1}{4\Phi(\rho)^{-3}-2-\cdots}}} \tag{23}$$

Substituting the series expansion for $\Phi(\rho)$, Eq. (5), into the terms of Eq. (23) and truncating those terms to linear order leads to a new continued fraction that can be compared with the continued fraction for the ratio of Bessel functions,[11]

$$\frac{J_k(x)}{J_{k-1}(x)}=\cfrac{1}{\cfrac{2k}{x}-\cfrac{1}{\cfrac{2(k+1)}{x}-\cfrac{1}{\cfrac{2(k+2)}{x}-\cdots}}} \tag{24}$$

We then expect that, for $|\rho|\ll 1$,

$$Z_1(\rho)\cong\frac{J_{x+1}(x)}{J_x(x)} \tag{25}$$

where $x=1/[-2i\langle I\rangle\rho+\langle I^2\rangle\rho^2]$. It can be shown[12] that, to leading order, the asymptotic behavior of Eq. (23) is identical to that of Eq. (25). From the properties of Bessel functions as $x\to\infty$, $Z_1(\rho)$ in the vicinity of $\rho=0$ is finally given by[12]

$$Z_1(\rho)=\begin{cases}1-c\langle I\rangle^{1/3}\,i^{-1/3}\,|\rho|^{1/3}+\cdots & \text{for}\quad \langle I\rangle\neq 0 \qquad (26a)\\ 1-c\langle I^2\rangle^{1/3}\,2^{-1/3}\,|\rho|^{2/3}+\cdots & \langle I\rangle=0 \qquad (26b)\end{cases}$$

where $c=2\pi(16/3)^{1/6}/\Gamma(1/3)^2=1.15723...$. [The same leading order for $Z_1(\rho)$ can be obtained in the continuum limit of Eq. (22).[12,16]]

Since the characteristic function is the Fourier transform of the probability density $p(m)$, we obtain the charge distribution by inversion. In the case where $\langle I\rangle>0$ (or $\langle I\rangle<0$), we have

$$p(m)\sim m^{-4/3}\qquad \text{for}\quad m\gg\langle I\rangle\quad (\text{or } m\ll\langle I\rangle) \tag{27a}$$

while in the case of $\langle I\rangle=0$, we have

$$p(m)\sim m^{-5/3}\qquad \text{for}\quad |m|\gg\langle I^2\rangle^{1/2} \tag{27b}$$

Condition (27a) shows a one-sided power law whose exponent agrees with that of the steady distribution for constant injection.[8] Condition (27b) gives a symmetric power law.

Next we consider the pair-creation injection case. From (8a) and (8b) it is easy to show that $Z_1(\rho)$ is given by the following continued fraction:

$$Z_1(\rho)=\cfrac{1}{4\Phi(\rho)^{-1}-2-\cfrac{1}{4\Phi(\rho)^{-2}-2-\cfrac{1}{4\Phi(\rho)^{-2}-2-\cdots}}} \qquad (28)$$

Equation (28) can be transformed to

$$Z_1(\rho)=\cfrac{1}{4\Phi(\rho)^{-1}-2-\cfrac{1}{4\Phi(\rho)^{-2}-2+\left[\cfrac{1}{Z_1(\rho)}-4\Phi(\rho)^{-1}+2\right]}} \qquad (29)$$

which is a quadratic equation for $Z_1(\rho)$. Solving Eq. (29), we get

$$Z_1(\rho)=1-2\langle I^2\rangle^{1/2}|\rho|+\cdots \qquad (30)$$

The corresponding charge distribution becomes

$$p_1(m)\sim m^{-2} \qquad \text{for} \quad |m|\gg\langle I^2\rangle^{1/2} \qquad (31)$$

This probability density is symmetric and the exponent is the same as for a Lorentzian.

In Figs. 2–4, we show the results of numerical simulations for the cases $\langle I\rangle>0$, $\langle I\rangle=0$, and pair-creation injection.

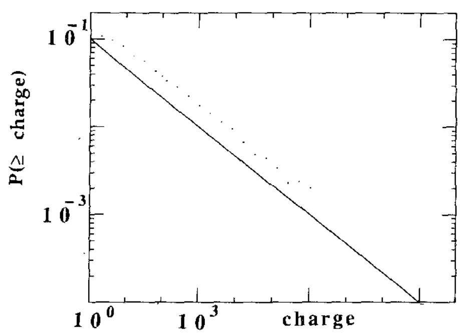

Fig. 2. The steady-state cumulative charge distribution in one dimension in the case of $\langle I\rangle\neq 0$. The straight line shows the theoretical slope $-1/3$. (The slope of the cumulative distribution is the slope of the original distribution increased by $+1$.)

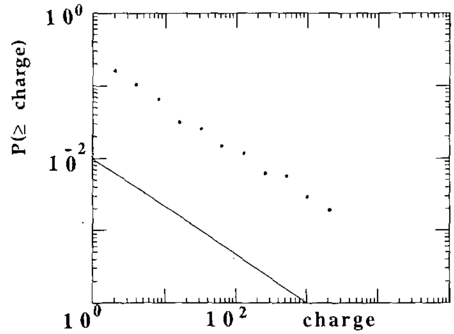

Fig. 3. The steady-state cumulative charge distribution in one dimension in the case of $\langle I \rangle = 0$. The straight line shows the theoretical slope $-2/3$.

For the mean-field case, the time-independent solution of Eq. (10) satisfies

$$Z_1(\rho) = \Phi(\rho)\, e^{Z_1(\rho) - 1} \tag{32}$$

In the vicinity of $\rho = 0$ we can expand $Z_1(\rho)$ around 1. Neglecting the higher-order terms of ρ, we get the solutions

$$Z_1(\rho) = \begin{cases} 1 - \sqrt{2}\, \langle I \rangle^{1/2}\, i^{-1/2}\, |\rho|^{1/2} + \cdots & \text{for} \quad \langle I \rangle \neq 0 \quad (33a) \\ 1 - \langle I^2 \rangle^{1/2}\, |\rho| + \cdots & \text{for} \quad \langle I \rangle = 0 \quad (33b) \end{cases}$$

The corresponding steady-state charge distribution in the case of $\langle I \rangle > 0$ (or $\langle I \rangle < 0$) becomes

$$p_1(m) \sim m^{-3/2} \qquad \text{for} \quad m \gg |\langle I \rangle| \tag{34a}$$

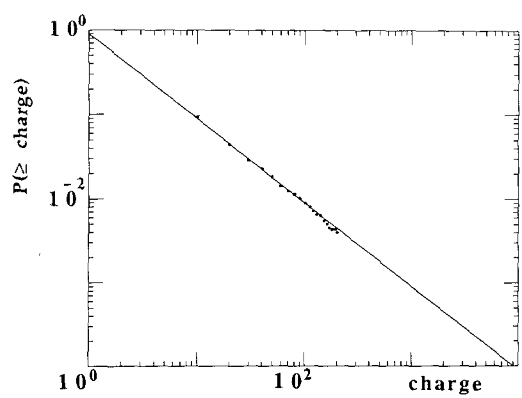

Fig. 4. The steady-state charge distribution in one dimension in the case of pair-creation injection. The straight line shows the theoretical slope -1.

Table I. The Steady-State Charge Distribution $p(m)$ and the Form of the Relaxation $g(t)$ for the One-Dimensional Case (1D) and the Mean Field case (MF)[a]

	$p(m)$	$g(m, t)$
1D, $\langle I \rangle \neq 0$	$m^{-4/3}$	$\exp[-t^3]$
1D, $\langle I \rangle = 0$	$m^{-5/3}$	$t^{-3/4}$
1D, pair creation	m^{-2}	t^{-1}
MF, $\langle I \rangle \neq 0$	$m^{-3/2}$	$\exp[-t^2]$
MF, $\langle I \rangle = 0$	m^{-2}	t^{-1}
MF, pair creation		

[a] In the case of $\langle I \rangle \neq 0$, $p(m)$ is one-sided, while in the other cases $p(m)$ is symmetric.

and in the case of $\langle I \rangle = 0$,

$$p_1(m) \sim m^{-2} \qquad \text{for} \quad m \gg \langle I^2 \rangle^{1/2} \tag{34b}$$

Again we have a one-sided power law in the case of $\langle I \rangle \neq 0$, and a symmetric one in the case of $\langle I \rangle = 0$. The results are summarized in Table I.

The results for the one-dimensional case can be understood intuitively as follows. As mentioned in Section 2, the charge of any particle is equal to the sum of the charges of the injected particles over its corresponding river basin. It is obvious from the evolution rule that the basin's left and right boundaries are simple random walks (see Fig. 5).

Therefore the area of the basin is roughly given by the product $S = hw$, where h and w denote the basin's height and width, respectively. Since the boundaries of the basin are random walks, w is proportional to $h^{1/2}$ and the

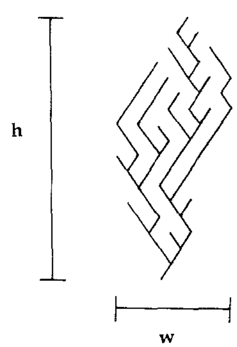

Fig. 5. An example of the space-time trajectory of a particle. The width and height are given by l and h, respectively.

distribution of the height h is identical to the distribution of a Brownian particle's recurrence time,[13]

$$p(h) \sim h^{-3/2} \tag{35}$$

Hence, the distribution function for the area of the basin, $S = h \cdot w \propto h^{3/2}$, is given by

$$p(S) = p(h) \frac{dh}{dS} \sim S^{-4/3} \tag{36}$$

By definition the charge of a particle having basin size S is given by the sum of S independent random variables I_j. In the case $\langle I \rangle \neq 0$ the charge m is obviously proportional to S, hence we have $p(m) \sim m^{-4/3}$. When the average value of I is zero, m becomes proportional to the variance $(\langle I^2 \rangle S)^{1/2}$. Substituting the relation $m \sim S^{1/2}$ into Eq. (36), we get $p(m) \sim m^{-5/3}$.

A pair creation inside the area does not contribute to the charge m, because positive and negative charges cancel. The contribution to m comes only from the boundaries, namely, m is given by the sum of $2h$ random variables I_j. In the case of pair-creation injection, the mean value of I_j is automatically zero, so m is proportional to $(2h)^{1/2}$. Substituting this relation into Eq. (35), we obtain the charge distribution $p(m) \sim m^{-2}$. These intuitive results agree perfectly with the analytic results of Eqs. (27a), (27b), and (31).

4.2. Uniqueness and Stability

In the preceding section we have seen that there exists a statistically-invariant steady state which is realized by injection. Here we show that the steady-state is uniquely determined for a given injection, independently of the initial conditions.

Let us introduce a perturbation $\tilde{Z}_r(\rho, t)$ around the steady state $Z_r(\rho)$:

$$Z_r(\rho, t) = Z_r(\rho) + \tilde{Z}_r(\rho, t) \tag{37}$$

Due to the linearity of Eq. (4), the equation for the perturbation $\tilde{Z}_r(\rho, t)$ in the one-dimensional case is given by

$$\tilde{Z}_r(\rho, t+1) = \tfrac{1}{4}\Phi(\rho)^r \, [\tilde{Z}_{r+1}(\rho, t) + 2\tilde{Z}_r(\rho, t) + \tilde{Z}_{r-1}(\rho, t)] \tag{38}$$

with the boundary conditions $\tilde{Z}_0(\rho, t) = 0$ and $\tilde{Z}_r(0, t) = 0$.

Taking the absolute value of both sides of Eq. (38) and replacing the right-hand side by the maximum of $\{|\tilde{Z}_{r'}(\rho, t)|,\ r' = r-1,\ r,\ r+1\}$, we obtain the following relation:

$$|\tilde{Z}_r(\rho, t)| < |\Phi^r(\rho)| \cdot \max\{|\tilde{Z}_{r'}(\rho, t-1)|, r' = r-1, r, r+1\} \tag{39}$$

Iterating Eq. (39) t times and using the inequality $|\Phi(\rho)| \leqslant 1$, which is valid for any characteristic function, we get for $r \neq 0$

$$|\tilde{Z}_r(\rho, t)| \leqslant |\Phi^r(\rho)|^t \cdot \max\{|\tilde{Z}_{r'}(\rho, 0)|, r' = 1, 2,...\} \tag{40}$$

Therefore for any ρ which satisfies $|\Phi(\rho)| < 1$, the perturbation goes to zero exponentially with time. This is true for any initial condition and it means that the steady state is unique and stable. The stability for the case $|\Phi(\rho)| = 1$ is not trivial, but as we show in the Appendix, even in this case the system relaxes to a unique steady state for any initial perturbation.

In the mean-field case (B), the equation for the perturbation is derived by substracting Eq. (32) from Eq. (10), namely

$$\tilde{Z}_1(\rho, t+1) = \frac{\Phi(\rho)}{e} \left[e^{Z_1^*(\rho) + \tilde{Z}_1(\rho, t)} - e^{Z_1^*(\rho)}\right] \tag{41}$$

Taking the absolute value on both sides of Eq. (39) and using the general inequality

$$|e^x - e^y| \leqslant e\,|x - y| \qquad \text{for} \quad |x|, |y| \leqslant 1$$

we find that

$$|\tilde{Z}_1(\rho, t)| < |\Phi(\rho)|^t \cdot |\tilde{Z}_1(\rho, 0)| \tag{42}$$

Again for $|\Phi(\rho)| < 1$ it is trivial that the perturbation vanishes as $t \to \infty$, so the steady-state solution $Z_1(\rho)$ is unique and stable. The discussion in the Appendix is also applicable to this case; therefore the uniqueness and stability hold for any size of injection.

4.3. Spatial Correlation

We have seen that the exponents of the steady-state distributions are different in the one-dimensional and in the mean-field cases (except for the case of pair-creation injection). This means that the spatial restriction in the one-dimensional case is strong enough to modify the exponent from the mean-field value. In this subsection we observe spatial correlations in the one-dimensional model and clarify the nature of spatial fluctuations.

Usually spatial correlation is observed by introducing a correlation function such as $\langle m(0, t) \cdot m(j, t) \rangle$. However, in our model such a function does not have significant meaning because it diverges in the steady state. Instead of using the correlation function, we can use the many-body characteristic function $Z_r(\rho)$ to see the nature of the singular fluctuation.

From the basic steady-state equation (22), we can show that $Z_r(\rho)$ can be expanded in the vicinity of $\rho = 0$ as

$$Z_r(\rho) = 1 - c_1 r\, |\rho|^\beta + \cdots \tag{43}$$

where c_1 is a constant and β is either 1/3 or 1/2, depending on the injection type. If the distribution was spatially independent, $Z_r(\rho)$ should be equal to $[Z_1(\rho)]^r$; however, it is easy to prove that these two expressions are identical only up to order ρ^β, and there is a nonvanishing term of order $\rho^{2\beta}$ as

$$Z_1(\rho)^r - Z_r(\rho) = \frac{r(r-1)}{2} (c_1)^2 \rho^{2\beta} + \cdots \neq 0 \tag{44}$$

Therefore we can conclude that there really exists spatial correlation. However, it is not easy to detect this correlation numerically, because it is higher order in ρ. Actually, both multifractal and R/S analysis, which are applicable to this model, are unable to numerically detect correlations.(14) It is an open problem to develop a numerical method which could confirm the existence of this delicate spatial correlation.

5. RELAXATION

In Section 4.2 we have seen that the steady state realized by injection is unique and stable. In this section we shall analyze the relaxation to the steady state.(15)

In the continuum space and time limit and (for small ρ), the evolution equation for the perturbation in the case of independent random injection in one-dimension [Eq. (38)] takes the form of the following partial differential equation:

$$\frac{\partial \tilde{Z}_r(\rho, t)}{\partial t} = D \frac{\partial^2 \tilde{Z}_r(\rho, t)}{\partial r^2} - Rr\tilde{Z}_r(\rho, t), \qquad \rho \to 0 \tag{45}$$

Here $D = 1/(4\Delta t)$, and $R = (-i\langle I\rangle \rho + \langle I^2\rangle \rho^2/2)/\Delta t$, where Δt denotes the time step. By taking the Fourier transform with respect to t [and using the boundary conditions $\tilde{Z}_0(\rho, t) = 0$ and $\tilde{Z}_\infty(\rho, t) = 0$], one can solve Eq. (45) in terms of Airy functions $Ai(x)$ as(16)

$$\tilde{Z}_r(\rho, t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} E_k(r) \tag{46}$$

where

$$\lambda_k = |a_k|\, (2DR^2)^{1/3}$$

and

$$E_k(r) = Ai((R/2D)^{1/3} r + a_k)$$

and a_k denotes the kth zero of the Airy function, $a_1 = -2.33...$, $a_2 = -4.08...$, etc. Since λ_1 is the smallest of $\{\lambda_k\}$, we have the following time dependence when $t \to \infty$ in the case of $\langle I \rangle \neq 0$:

$$\tilde{Z}_1(\rho, t) = f_1(\rho) \exp\{-b_1 \langle I \rangle^{2/3} [\exp(-\pi i/3)] \rho^{2/3} t\} \tag{47}$$

where $b_1 = |a_1| \, 2^{-2/3}/\Delta t$, and $f_1(\rho)$ is a function independent of t. In this type of process the relaxation is faster for larger values of ρ and slower for smaller values of ρ. The relaxation form of the corresponding mass distribution function can be obtained by taking the inverse Fourier transform of Eq. (47). Analytical expression of the inverse Fourier transform can be obtained by applying the theory of stable distributions. [Note that the time-dependent factor on the right-hand side of Eq. (47) has the same form as the characteristic function of the stable distribution, $p(x, 2/3, -2/3)$.[2,13,17]] The time evolution of the perturbation is described by the following form:

$$\tilde{p}(m, t) = \int dm' \, \tilde{p}(m - m', 0) \, g(m', t) \tag{48}$$

Here the Green function $g(m, t)$ is given by

$$g(m, t) \sim \frac{1}{x} e^{-16/(27x^2)} W_{1/2, 1/6}\left(\frac{32}{27x^2}\right) \tag{49}$$

where $x = b_1 \, |\langle I \rangle / m|^{2/3} \, t$ and $W_{\mu,\nu}(z)$ is the Whittaker function. The asymptotic behavior of Eq. (49) is estimated as

$$g(m, t) \sim \exp\left(-\frac{b_1 \, |\langle I \rangle|^2 \, t^3}{m^2}\right) \tag{50}$$

This decay is faster than Gaussian and its characteristic time is proportional to $m^{2/3} \, |\langle I \rangle|^{2/3}$.

In the case of $\langle I \rangle = 0$, Eq. (47) for $\rho \to 0$ is replaced by

$$\tilde{Z}_1(\rho, t) = f_2(\rho) \exp\{-b_2 \langle I^2 \rangle^{2/3} \rho^{4/3} t\}, \qquad t \to \infty \tag{51}$$

where $b_2 = |a_1| \, 2^{-4/3}/\Delta t$, and f_2 is independent of t. The Fourier inversion of Eq. (51) has the form of Eq. (48) where the Green function is given by a symmetric stable distribution with the characteristic exponent 4/3. For large $|m|$ the asymptotic behavior is estimated as

$$g(m, t) \sim \langle I^2 \rangle^{-1/2} \, t^{-3/4} \qquad \text{for} \quad t \gg m^{4/3} \langle I^2 \rangle^{-2/3} \tag{52}$$

This power-law decay is slow enough to be confirmed numerically, as shown in Fig. 6. The points scatter at large t, but they are in good agreement with the theoretical slope.

The reason for these two different types of decay, Eq. (50) and Eq. (52), will be discussed at the end of this section.

In the case of pair-creation injection, the perturbation equation in the continuum limit is derived from Eq. (8b) in the vicinity of $\rho \to 0$:

$$\frac{\partial \tilde{Z}_r(\rho, t)}{\partial t} = \frac{1}{4\Delta t} \frac{\partial^2 \tilde{Z}_r(\rho, t)}{\partial r^2} - b_3 \rho^2 \tilde{Z}_r(\rho, t) \tag{53}$$

where $b_3 = \langle I^2 \rangle / \Delta t$. The solution of Eq. (53) is given by

$$\tilde{Z}_r(\rho, t) = \frac{\exp(-b_3 \rho^2 t)}{(\pi t/\Delta t)^{1/2}} \int_0^\infty dr' \, \tilde{Z}_{r'}(\rho, 0) \times \left\{ \exp\left[-\frac{(r-r')^2}{t/\Delta t} \right] - \exp\left[-\frac{(r+r')^2}{t/\Delta t} \right] \right\} \tag{54}$$

The Fourier inversion of Eq. (54) can be put into the form of Eq. (48), and the corresponding Green function has long-time behavior

$$g(m, t) \sim \langle I^2 \rangle^{-1/2} t^{-1} \qquad \text{for} \quad t \gg m^2 \langle I^2 \rangle^{-1} \tag{55}$$

We again have a power-law decay in this case, but with a different exponent than in the case of random zero-mean injection.

In the mean-field case, the equation for small perturbations is given by the linearized version of Eq. (41):

$$\tilde{Z}_1(\rho, t+1) = \Phi(\rho) \, e^{Z_1(\rho) - 1} \tilde{Z}_1(\rho, t) \tag{56}$$

Substituting the steady-state solutions, Eqs. (33a) and (33b), we have the following solutions for small ρ:

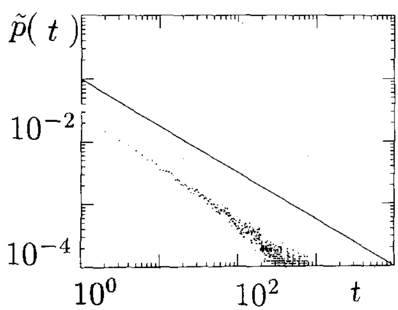

Fig. 6. The relaxation function in the case of $\langle I \rangle = 0$ plotted on a log-log scale. The straight line shows the theoretical slope, $-3/4$.

In the case of $\langle I\rangle \neq 0$,

$$\tilde{Z}_1(\rho, t) \cong \tilde{Z}_1(\rho, 0) \exp[-\sqrt{2}\, \langle I\rangle^{1/2}\, i^{-1/2}\, |\rho|^{1/2}\, t], \qquad \rho \to 0 \quad (57a)$$

In the case of $\langle I\rangle = 0$,

$$\tilde{Z}_1(\rho, t) \cong \tilde{Z}_1(\rho, t) \exp[-\langle I^2\rangle^{1/2}\, |\rho|\, t], \qquad \rho \to 0 \quad (57b)$$

Corresponding temporal behavior of the distribution functions is also given in the form of Eq. (48). The asymptotic behavior of the corresponding Green functions in the continuum limit is obtained as follows.

In the case of $\langle I\rangle \neq 0$,

$$g(m, t) \sim \exp\left[-\frac{|\langle I\rangle|\, t^2}{2m}\right], \qquad t \to \infty \quad (58a)$$

In the case of $\langle I\rangle = 0$,

$$g(m, t) \sim \langle I^2\rangle^{-1/2}\, t^{-1}, \qquad t \gg m\langle I^2\rangle^{-1/2} \quad (58b)$$

We again have a fast decay (Gaussian decay) in the case of biased injection and a slow decay in the case of zero-mean injection. The results are summarized in Table I together with the corresponding steady-state distributions.

The reason for the rapid decay in the biased injection cases ($\langle I\rangle \neq 0$) can be understood intuitively as follows. Assume that we have perturbed the steady-state distribution by introducing a sharp peak in the charge distribution around m_0. The number of particles of charge m_0 decreases by aggregation or by injection. An aggregation event of two particles with charges m_0 and m creates a particle of charge $m + m_0$. Consequently, the aggregation process scatters the distribution around m_0 continuously. On the other hand, injection shifts the charge of the particle from m_0 to $m_0 + I$. In the case of positive-mean injection, for example, the injection causes the peak of the charge distribution to drift continuously toward larger m, and aggregation also scatters m_0 to larger values. Thus the relaxation observed at a fixed charge m_0 is very fast. On the other hand, in the case of $\langle I\rangle = 0$ (including pair-creation injection), the perturbation diffuses slowly around m_0 due to the injection and scatters nearly symmetrically by aggregation. The peak of the perturbation does not move on average, and so the decay is much slower than in the biased-injection case.

6. DISCUSSION

In this paper we analyzed the statistical properties of an aggregation system, for one-dimensional and mean-field cases, using the equation for the many-body characteristic function. In the case of no injection, the particle number decays following a power law. But there appears a nontrivial statistically-invariant steady state if we supply particles at a constant rate. The most interesting point here is that the system converges to the steady state even in the absence of a sink or particle breakup. It is easy to confirm from Eqs. (4) and (10) that the variance of particle charge increases linearly with time, namely, it diverges in the limit of $t \to \infty$. Usually such divergence means that the system does not have a stable state, but in this case the charge distribution surely converges to a power-law distribution which has infinite mean value and variance.

According to the theory of stable distributions,[13] the divergence of the variance is essential to power law distributions. The central limit theorem, though, can be generalized to include independent random variables with divergent variances. The limiting distribution of the sum of such variables is, if it exists, a non-Gaussian stable distribution with a long power-law tail. In our model, the variance is divergent in the steady state and we can decompose the particle charge into a sum of aggregate variables. We cannot directly apply the generalized central-limit theorem because we know that there exists a weak spatial correlation among the random variables. However, it seems likely that the charge distribution here belongs to the domain of attraction of a non-Gaussian stable distribution and thus has a power-law tail.

Our aggregation model does not have any control parameter and automatically assumes a power-law distribution, typically a sign of critical behavior. In order to see the criticality from the point of view of a branching model, one just needs to invert the time axis.[2] In the branching model the system can be described by the values of the branching probability. For high branching probability the number of particles increases exponentially and for low branching probability it decreases exponentially. There exists a critical value of the branching probability at which the particle number becomes constant constant on average. This model is equivalent to one of the stochastic cellular automata analyzed by Domany and Kinzel,[18] and it is known that these phase transitions are similar to that of directed percolation. In our aggregation model, the number of particles is kept constant by injection; therefore, the corresponding branching probability must be equal to the critical value to keep the particle number constant.

In Sections 2, 4, and 5 we addressed the question of critical dimension. It is an open problem to analyze our model in dimensions greater than one. Above the critical dimension, the mean-field model gives the correct exponents for both the steady state and for the relaxation. It should be noted that in the case of pair creation, the exponents for one dimension coincide with those for the mean field. This result seems to suggest that the critical dimension is one in the case of pair-creation injection. On the other hand, for random injection (both for $\langle I \rangle \neq 0$ and for $\langle I \rangle = 0$), the exponents for one dimension are different than the mean-field values, which indicates that the critical dimension is greater than one. It is therefore likely that the critical dimension is not universal, but depends on the type of injection.

As we showed in Section 4, the steady-state power-law distribution is very robust. It is uniquely determined by the type of injection and is independent of the specific details of the injection or initial conditions. The functional form of the relaxation to the steady state is also determined by the type of injection. We have considered only injection distributions with finite variance. It is an open problem to analyze the case of injection distributions with divergent variance.

APPENDIX

The characteristic function $\Phi(\rho)$ by definition satisfies one of the following three criteria[19]:

(i) $|\Phi(\rho)| < 1$ for all ρ, except for $\rho = 0$.

(ii) $|\Phi(\rho)| = 1$ for all ρ.

(iii) $|\Phi(\rho)| = 1$ for countably many ρ.

The second case (ii) corresponds to a completely degenerate distribution, i.e., the distribution function is represented by a δ-function,

$$\rho(I) = \delta(I - I_0) \tag{A1}$$

The third case corresponds to distributions degenerate at periodic points, i.e.,

$$p(I) = \sum_{j=-\infty}^{\infty} a_j \delta(I - jI_0 - I_1) \tag{A2}$$

where I_0 and I_1 are constants and $\{a_j\}$ satisfies the normalization condition $\sum_j a_j = 1$ and $a_j \geqslant 0$. All other distributions belong to the first case (i).

From Eq. (40) [or Eq. (42)], we have the following inequality for the perturbation:

$$|\tilde{Z}(\rho, t)| \leqslant |\Phi(\rho)|^t \cdot \text{const} \tag{A3}$$

In case (i), it is trivial that $|\tilde{Z}(\rho, t)| \to 0$ for all values of ρ as $t \to \infty$ [remember that for $\rho = 0$, $\tilde{Z}(\rho, t) = 0$ from the boundary condition]. In case (iii), obviously $|\tilde{Z}(\rho, t)| \to 0$ as $t \to \infty$ for all ρ except for the points $\{2\pi j/I_0,\ j=0, \pm 1, \pm 2,...\}$. Since a characteristic function is a continuous function, $|\tilde{Z}(\rho, t)|$ must vanish for all ρ in the limit $t \to \infty$.

In case (ii), we can assume that $I_0 > 0$ without loss of generality. The uniqueness and stability can be proven easily if we modify the definition of the characteristic function as

$$Z(\rho, t) = \langle e^{-\rho m} \rangle, \qquad \Phi(\rho) = e^{-\rho I_0} \tag{A4}$$

In the same way as we derived Eqs. (38) and (40), we get the following inequality:

$$|\tilde{Z}(\rho, t)| \leqslant e^{-\rho I_0 t} \cdot \text{const} \tag{A5}$$

and this shows that $|\tilde{Z}(\rho, t)| \to 0$ as $t \to \infty$ for all values of ρ.

Thus the perturbation $|\tilde{Z}(\rho, t)|$ vanishes in the limit $t \to \infty$ in all cases. In other words, the aggregation system converges to the steady state for any initial disturbance on the power-law distribution.

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